How to compute SVD of a matrix g?

Example: $g = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Key idea: consider gg^T and g^Tg.

(Both are symmetric, hence can apply spectral theorem to find orthonormal eigenbasis).

Suppose (v_i) is an such a basis of g^Tg, with eigenvalues λ_i.

Then, $gg^T(gv_i) = g\lambda_i v_i = \lambda_i(gv_i)$

Also, $\|gv_i\|^2 = v_i^T g^T g v_i = \lambda_i \geq 0$

So, let $\sigma_i = \sqrt{\lambda_i}$,

$$u_i = \frac{gv_i}{\sigma_i} \text{ for non-zero } \sigma_i.$$

Note that for $i \neq j$,

$$\langle u_i, u_j \rangle = \frac{1}{\sigma_i \sigma_j} v_j^T g^T g v_i$$

$$= \frac{1}{\sigma_i \sigma_j} \lambda_i v_j^T v_i = 0.$$

So, $\begin{pmatrix} - & \vec{u}_1^T & - \\ - & \vec{u}_2^T & - \\ & \vdots & \\ - & \vec{u}_m^T & - \end{pmatrix} g \begin{pmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_n \\ | & & | \end{pmatrix}$

$$= \begin{pmatrix} \sigma_1 & & & & & \\ & \ddots & & & & \\ & & \sigma_m & & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} | & | & & | \\ \vec{u}_1 & \vec{u}_2 & \cdots & \vec{u}_m \\ | & | & & | \end{pmatrix} \begin{pmatrix} \sigma_1 & & & & & \\ & \ddots & & & & \\ & & \sigma_m & & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \end{pmatrix} \begin{pmatrix} - & \vec{v}_1^T & - \\ & \vdots & \\ - & \vec{v}_n^T & - \end{pmatrix}$$

$$= g.$$

1. (**20pts**) This question is about singular value decomposition.

   (a) Consider

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

   i. Compute $A^T A$. Find the eigenvalues of $A^T A$.

   ii. Compute the singular value decomposition of A.

   iii. Write A as a linear combination of eigen-images.

   (b) Hence or otherwise, compute the singular value decomposition of

$$A' = \begin{pmatrix} 2 & 2 & 0 \\ 2 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix},$$

   and write A' as a linear combination of eigen-images.

i. $A^T A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$$= \begin{pmatrix} 2 & 2 & 0 \\ 2 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The characteristic polynomial of $A^T A$ is

$$p(\lambda) = \det\left(A^T A - \lambda I\right)$$

$$= \det \begin{pmatrix} 2-\lambda & 2 & 0 \\ 2 & 2-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{pmatrix}$$

$$= (1-\lambda)\left((\lambda-2)^2 - 4\right)$$

$$= (1-\lambda)\,\lambda\,(\lambda-4)$$

$p(\lambda) = 0 \Rightarrow \lambda = 4,\ 1,\ \text{or } 0.$

ii. We first compute an orthonormal eigenbasis for A^TA.

They are $v_1 = \begin{pmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \\ 0 \end{pmatrix}$ $(\lambda_1 = 4)$

$v_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ $(\lambda_2 = 1)$

$v_3 = \begin{pmatrix} \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} \\ 0 \end{pmatrix}$ $(\lambda_3 = 0)$

Then, $u_1 = \dfrac{A v_1}{\sqrt{\lambda_1}} = \begin{pmatrix} \sqrt{2} \\ \sqrt{2} \\ 0 \end{pmatrix} \cdot \frac{1}{2} = \begin{pmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \\ 0 \end{pmatrix}$

$u_2 = \dfrac{A v_2}{\sqrt{\lambda_2}} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

(No need to compute u_3 by $\frac{A v_3}{\sqrt{\lambda_3}}$ since $\lambda_3 = 0$).

Extend $\{u_1, u_2\}$ to an orthonormal basis,

that is $\left\{ u_1, u_2, \begin{pmatrix} \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} \\ 0 \end{pmatrix} \right\}$.

So, $A = \begin{pmatrix} | & | & | \\ u_1 & u_2 & u_3 \\ | & | & | \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} - & v_1^T & - \\ - & v_2^T & - \\ - & v_3^T & - \end{pmatrix}$

$$= \begin{pmatrix} \frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & 0 & -\frac{\sqrt{2}}{2} \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 \end{pmatrix}$$

iii. $A = \begin{pmatrix} \frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & 0 & -\frac{\sqrt{2}}{2} \\ 0 & 1 & 0 \end{pmatrix} \cdot \left(2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right)$

$$\begin{pmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 \end{pmatrix}$$

$$= 2\begin{pmatrix} \frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & 0 & -\frac{\sqrt{2}}{2} \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 \end{pmatrix}$$

$$+ \begin{pmatrix} \frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & 0 & -\frac{\sqrt{2}}{2} \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 \end{pmatrix}$$

$$= 2\begin{pmatrix} \frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & 0 & -\frac{\sqrt{2}}{2} \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} \frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & 0 & -\frac{\sqrt{2}}{2} \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$= 2\begin{pmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \\ 0 \end{pmatrix}\begin{pmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(b) Note that

$$A' = 2\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + 3\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= 4\begin{pmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \\ 0 \end{pmatrix}\begin{pmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \end{pmatrix} + 3\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 \end{pmatrix}$$

So, $$A' = \begin{pmatrix} \frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & 0 & -\frac{\sqrt{2}}{2} \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 4 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 \end{pmatrix}$$

Remark: general method to extend a linearly independent set $\{v_1, \cdots, v_m\}$ $(m < n)$ to an orthonormal basis of $\mathbb{R}^n$:

First, we extend $\{v_1, \cdots, v_m\}$ to a basis.

Denote $\begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix}$ -i-th by e_i,

and $A = \{v_1, \cdots, v_m\}$.

starting from 1 to n,

if $e_i \notin \operatorname{span}(A)$,

add e_i to the set A.

If there are n elements in A,

then stop the iteration.

After that, we obtain a basis A.

Then, apply Gram-Schmidt orthogonalization process to obtain an orthonormal basis.